Comparison of Monte Carlo and Statistical Treatments of Heat-Transfer Data Uncertainties

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Statistical error analysis (root-sum-square) is compared to the Monte Carlo method for the treatment of data uncertainties in heat transfer through an aerospace structure. Comparison is made on the basis of numerical results. Numerical results for the two methods are nearly identical. The root-sum-square method also yields useful intermediate results. The intermediate results are useful for determining which variables are critical.

Nomenclature

A_{ij} = area for conduction between nodes i and j
$\mathfrak{F}_{ij}$ = radiation exchange factor between nodes i and j
G = Gaussian random number
k = thermal conductivity
Q = rate of heat transfer
R = thermal resistance
T = temperature
U = uniformly distributed random number
x_{ij} = conduction path length between nodes i and j
$\bar{x}$ = mean value of random independent variable
x_i = random value for independent variable
σ = Stefan-Boltzmann constant or standard deviation

Introduction

THE Monte Carlo method has been applied to the treatment of data uncertainties in the analysis of heat transfer through an aerospace structure by Howell.[1] Ishimoto and Bevans[2] have applied statistical error analysis to a spacecraft thermal model. In this paper, both techniques are applied to the same problem so that a direct comparison of the techniques is possible. In particular, in this paper, it will be shown that the root-sum-square (rss) technique of statistical error analysis yields results comparable to the Monte Carlo method, together with additional information about the sensitivity of results to uncertainties in independent variables and an indication of the importance of each uncertainty (identification of critical variables). For this work, we have chosen a one-dimensional, steady-state, fixed-boundary-condition problem with constant thermal conductivities and emissivities. However, the techniques for investigating the effect of data uncertainties on computed heat-transfer rates can be applied equally well to the more general three-dimensional transient case involving both time-varying boundary conditions and temperature-variable thermal conductivities and emissivities.

The structure chosen for analysis, 20 in. wide × 12 in. deep, is shown in Fig. 1. It represents a hydrogen Dewar with an external boundary temperature of 1460°R and with liquid hydrogen slightly above its normal boiling point at the internal boundary. It is actually a section of the structure investigated by Howell,[1] this section being easier to analyze than that of Howell and the results being just as instructive. The external layer of densified quartz fiber is utilized in an attempt to bring the fiberglass honeycomb sandwich down to a temperature at which it can survive. Both the fiberglass honeycomb and the space between it and the inner structure are under vacuum. The structure (0.070-in. aluminum) offers negligible thermal resistance. The thermal conductivity of the foam mounted inside the tank is affected considerably by the temperature of the hydrogen gas within its pores, decreasing drastically as the hydrogen/foam interface is approached.

It can be shown that, except for the honeycomb cross-sectional (solid) area and depth, the effect of uncertainties in areas and lengths on heat transfer is negligible, the products of uncertainties and sensitivity coefficients being negligibly small. As a consequence, in this work we have concentrated on uncertainties in thermal conductivity and in the radiation exchange factor. Values for random variables together with their standard deviations are shown in Table 1, where units on thermal conductivity are Btu-ft/hr-ft²-°F. In general, thermal conductivities are a function of temperature, but for this work they were assumed constant between nodes. The values listed in Table 1 come, for the most part, from Ref. 1.

A number of additional simplifying assumptions have been made:

1) All independent random variables are assumed to be Gaussian.
2) The hot-side wall temperature is a constant 1460°R.
3) The cold-side wall temperature is a constant 37°R.
4) Interface resistance to heat transfer is negligible.
5) Surface emissivities do not vary with temperature.
6) There is no radiation between nodes 2 and 4.

Other assumptions may be inferred from the manner in which calculations are done.

Nominal Solution

To determine the heat-transfer rate through the structure of Fig. 1, a suitable algorithm must be devised for simultaneously satisfying

$$Q_r = \sigma \mathfrak{F}_{56} A_{56} (T_5^4 - T_6^4) \tag{1}$$

Table 1 Random variables

Random variable	Nominal value	Standard deviation
k_{01}	0.040	0.005
k_{12}	0.22	0.050
k_{23}	0.22	0.050
k_{34}	0.22	0.050
k_{45}	0.22	0.050
k_{67}	50.	5.000
k_{78}	0.042	0.005
k_{89}	0.030	0.005
k_{910}	0.020	0.002
k_{1011}	0.012	0.002
A_{23}	0.023 ft²	0.002
A_{34}	0.023 ft²	0.002
x_{23}	0.4375 in.	0.004375
x_{34}	0.4375 in.	0.004375
$\mathfrak{F}_{56}$	0.6364	0.04242

Presented as Paper 75-710 at the AIAA 10th Thermophysics Conference, Denver, Colo., May 27-29, 1975; submitted Sept. 30, 1976; revision received April 26, 1977.
Index category: Thermal Modeling and Analysis.
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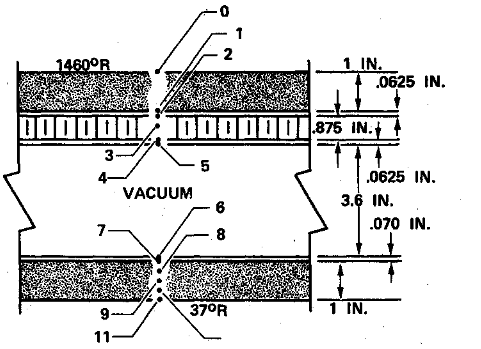

Fig. 1 Spacecraft hydrogen Dewar wall construction.

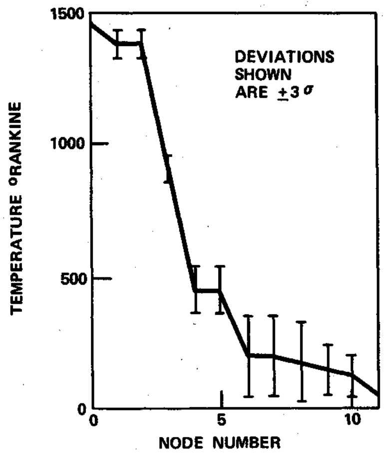

Fig. 2 Monte Carlo solution for temperature distribution.

$$Q_c = \frac{(T_0 - T_5) + (T_6 - T_{11})}{\sum_{0,1}^{4,5} R_{ij} + \sum_{6,7}^{10,11} R_{ij}} \tag{2}$$

$$Q_c = Q_r \tag{3}$$

where

$$R_{ij} = x_{ij} / k_{ij} A_{ij} \tag{4}$$

Utilizing the nominal values of Table 1, the nominal solution is 65.1 Btu/hr for an area of 1.6667 ft^2.

Monte Carlo Solution

The Monte Carlo technique consists of obtaining many solutions to a given problem, each based on a unique set of randomly determined values for the independent variables. Even if, unlike our case, the random variables are not Gaussian, the dependent variables will tend to have a Gaussian distribution as a consequence of the central limit theorem.[5] As the number of replications becomes large, the average value will approach the nominal value of the preceding section. In addition, the standard deviations of the dependent variables allows us to see quantitatively the effect of uncertainties in data.

Random variables in this work were obtained from

$$x_i = \bar{x} + G\sigma_x \tag{5}$$

where G is a Gaussian random number with a mean of zero and a standard deviation of unity obtained from

$$G = \sum_{1}^{12} U_i - 6.0 \tag{6}$$

where U_i is a uniformly distributed random number. Uniformly distributed random numbers are obtained easily from a computer routine, which utilizes the power-residue method.[3] More recently devised schemes[4] for obtaining Gaussian random numbers are more economical.

Some input values are not random but rather are correlated with other input values. For example, if the thermal conductivity between nodes 1 and 2, k_{12}, is 10% above nominal, then k_{23}, k_{34}, and k_{45} also will be 10% above nominal. Such correlations must be taken into account when determining the array of values to be used for each replication of the Monte Carlo solution. In our problem, correlations also exist between x_{23} and x_{34} and between A_{23} and A_{34}.

For 100 replications, mean values and three standard deviations for temperature are shown in Fig. 2. The heat-transfer rate is computed to be 64.4±13.8 Btu/hr for 1.6667 ft^2. It is observed that the nominal temperature of the fiberglass honeycomb outer facesheet, 918°F, exceeds the capability of even polyimide materials and that this temperature has a probability of 0.5 of being exceeded. And, for example, the $+1.28\ \sigma$ value (943°F) still will be exceeded 10

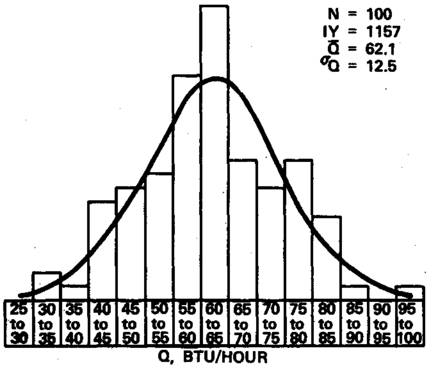

Fig. 3 Histogram for Q resulting from 100 replications.

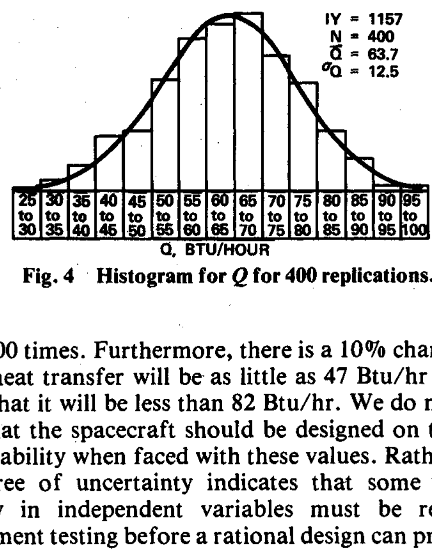

Fig. 4 Histogram for Q for 400 replications.

out of 100 times. Furthermore, there is a 10% chance that the rate of heat transfer will be as little as 47 Btu/hr and a 90% chance that it will be less than 82 Btu/hr. We do not mean to imply that the spacecraft should be designed on the basis of 0.9 probability when faced with these values. Rather, perhaps the degree of uncertainty indicates that some of the uncertainty in independent variables must be removed by development testing before a rational design can proceed.

The quality of data resulting from Monte Carlo simulation is affected to some extent by the number of replications: the more, the better. Figures 3 and 4 show a comparison of histograms and appropriate Gaussian probability density curves for 100 and 400 replications. It is seen that the fit for 100 replications is only fair, whereas that for 400 replications is excellent. However, the values for $\bar{Q}$ and σ_Q vary only slightly.

Figures 5 and 6 show how $\bar{Q}$ and σ_Q vary with number of replications for the particular set of random numbers utilized in this study. It would be expected that a different set of random numbers would give a slightly different set of results. In any event, as the number of replications becomes very large, the mean value should approach the nominal value of 65.1 Btu/hr computed initially. It is seen in Fig. 5 that increasing the number of replications does not have such an

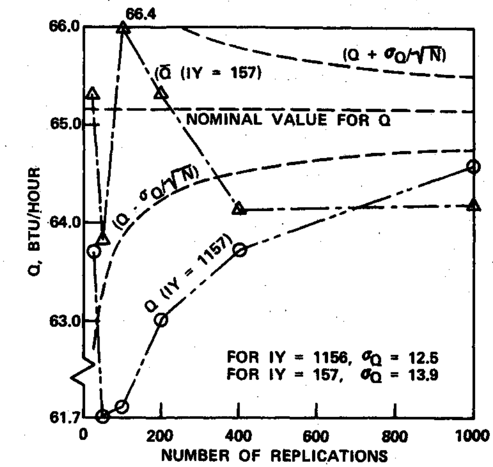

Fig. 5 Arithmetic mean for Q vs number of replications.

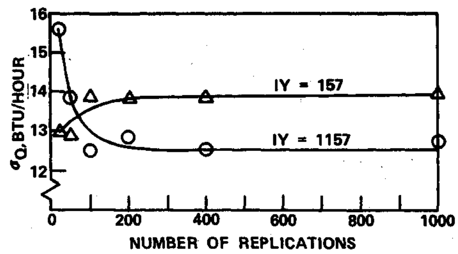

Fig. 6 Effect of number of replications on standard deviation for Q.

effect. Figure 5 also shows the effect of starting the random number generator with a different value for the integer IY. In Fig. 6, we see that there is not much change in the value of σ after 100 replications. How many replications are right for a given problem depends upon the circumstances. In this case, 1000 replications were relatively inexpensive, although it appears that 100 replications were sufficient.

Statistical Error Analysis

Statistical error analysis is an alternative technique that can offer considerable insight and may offer computational economies as well. The basis for statistical error analysis is covered in a great many places, and, among these, Taylor's description[5] seems particularly lucid. Briefly, if the independent variables have a Gaussian distribution and partial derivatives of dependent variables with respect to independent variables can be considered linear over the range of uncertainty in the independent variables, then the standard deviation can be obtained merely by taking the square root of the sum of the squares of partial derivatives times the standard deviations (the rss technique):

$$\sigma_y = \left[\sum_{1}^{n} \left(\frac{\partial y}{\partial x_i} \sigma_{x_i} \right)^2 \right]^{1/2} \tag{7}$$

Taylor[5] is enthusiastic about the technique because, even though requirements for its application are seldom completely met, "...RSS error estimates can be close to the truth even when the random error sources do not meet all or even most of the restrictions..."

As in Monte Carlo simulation, particular attention must be paid to correlations among the input variables. Taylor[5] states that "...the standard deviation of the output error can be determined (for two variables) from the relation..."

$$\sigma_y = \left[\left(\frac{\partial y}{\partial x_1} \right)^2 \sigma_{x_1}^2 + \left(\frac{\partial y}{\partial x_2} \right)^2 \sigma_{x_2}^2 + 2\rho_{x_1,x_2} \left(\frac{\partial y}{\partial x_1} \right) \left(\frac{\partial y}{\partial x_2} \right) \sigma_{x_1} \sigma_{x_2} \right]^{1/2} \tag{8}$$

Table 2 Intermediate results from rss program

x_i	σ_{x_i}	$\dfrac{\partial Q}{\partial x_i}$	$\sigma_{x_i} \dfrac{\partial Q}{\partial x_i}$	$\left(\sigma_{x_i} \dfrac{\partial Q}{\partial x_i}\right)^2$
k_{01}	0.005	116.148	0.581	0.338
k_{12}	0.050	0.241	0.012	0.000
k_{23}	0.050	121.715	6.086	37.039
k_{34}	0.050	121.715	6.086	37.039
k_{45}	0.050	0.241	0.012	0.000
k_{67}	5.0	0.000	0.000	0.000
k_{78}	0.005	2.198	0.011	0.000
k_{89}	0.005	4.311	0.022	0.000
k_{910}	0.002	9.747	0.019	0.000
k_{1011}	0.002	26.989	0.054	0.003
A_{23}	0.002	1164.246	2.328	5.420
A_{34}	0.002	1164.246	2.328	5.420
x_{23}	0.004375	−61.203	0.268	0.072
x_{34}	0.004375	−61.203	0.268	0.072
$\mathfrak{F}_{56}$	0.04242	0.403	0.017	0.000

Table 3 Comparison between Monte Carlo and root-sum-square results

y_i	σ_{y_i} (Monte Carlo)	σ_{y_i} (root-sum-square)
Q	13.8 Btu/hr	13.1 Btu/hr
T_0	0.0°R	0.0°R
T_1	19.4°R	18.9°R
T_2	19.4°R	18.9°R
T_3	10.3°R	8.9°R
T_4	30.1°R	25.7°R
T_5	30.1°R	25.7°R
T_6	52.6°R	54.5°R
T_7	52.6°R	54.5°R
T_8	47.7°R	50.5°R
T_9	38.9°R	43.9°R
T_{10}	28.6°R	31.2°R
T_{11}	0.0°R	0.0°R

where ρ_{x_1,x_2} is the correlation coefficient. Furthermore, he states that "...the correlation coefficient is...usually 0 or ±1 in error analysis...." With a correlation coefficient of unity, Eq. (8) becomes

$$\sigma_{y'} = \left(\frac{\partial y}{\partial x_1} \right) \sigma_{x_1} + \left(\frac{\partial y}{\partial x_2} \right) \sigma_{x_2} \tag{9}$$

and the generalization is obvious. Equation (9), which accounts for correlation only, should not be confused with Eq. (7), which describes the rss technique.

Table 2 shows the variables, their standard deviations, the numerically determined values for $\partial Q/\partial x_i$, the product of σ_{x_i} and $\partial Q/\partial x_i$, and the square of the product. The values in the last column can be particularly informative. A relatively large value identifies a critical variable. Small values are associated with variables having either a small sensitivity coefficient or a small uncertainty. Obviously, research effort should be spent on critical variables rather than on the latter.

Table 3 compares the standard deviations in Q and temperatures obtained with the two methods. Agreement is surprisingly good. In some cases, uncertainties in temperatures by the rss technique were slightly larger than for the Monte Carlo technique. Since uncertainties in independent variables were Gaussian, partial derivatives must not have been sufficiently linear, and Monte Carlo values must be accepted as the more correct of the two.

Conclusions

We realize that uncertainties have been treated statistically in the aerospace field for many years, particularly in the guidance field. However, Howell[1] claims to be the first to apply the Monte Carlo simulation to aerospace heat transfer. It appears that Ishimoto and Bevans[2] were the first to apply statistical error analysis to aerospace heat transfer. The purpose of this paper is to compare the two methods by applying them to the same problem. The results show that the rss technique has the virtue of producing additional data, which can lead to the identification of critical variables. On the other hand, the Monte Carlo approach was more straightforward for programming. It appears that either technique is satisfactory for the statistical treatment of data uncertainties in aerospace heat transfer.

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